fourier transform

fourier Series → fourier Integral → fourier Transform

* **Complex fourier transform:-**

Let $F(x)$ be a function of class $L'(-\infty, \infty)$. Then complex fourier transform of function F is denoted by $\mathcal{F}[F(x)]$ or $f(p)$ and defined by

$$\mathcal{F}[F(x)] = \int_{x=-\infty}^{\infty} F(x)\, e^{ipx}\, dx = f(p)$$

where p is known as parameter

Inverse fourier transform of function $f(p)$ is defined by -

$$\mathcal{F}^{-1}[f(p)] = \frac{1}{2\pi}\int_{p=-\infty}^{\infty} f(p)\, e^{-ipx}\, dp.$$

Sine fourier Transform:-

Let $F(x)$ be a function of class $L'(0, \infty)$. Then sine fourier transform of F is denoted by

$$\mathcal{F}_s[F(x)] \text{ or } f_s(p).$$

and defined by -

$$\mathcal{F}[F(x)] = f(p) = \int_{x=0}^{\infty} F(x) \sin px \, dx.$$

Inverse sine fourier transform of function $f_s(p)$ is defined by -

$$\mathcal{F}_s^{-1}[f(p)] = \frac{2}{\pi} \int_{p=0}^{\infty} f_s(p) \sin px \, dp.$$

Cosine fourier transform :—

Let $F(x)$ be a function of class $L'(0,\infty)$. Then cosine fourier transform of F is denoted by -

$$\mathcal{F}_c[F(x)] \text{ or } f_c(p).$$

and defined by -

$$\mathcal{F}_c[F(x)] = f_c(p) = \int_{x=0}^{\infty} F(x) \cos px \, dx.$$

$$\mathcal{F}_c^{-1}[f_c(p)] = \frac{2}{\pi} \int_0^{\infty} f_c(p) \cos px \, dp.$$

Q. Find fourier transform of the following.

$$F(x) = \begin{cases} e^{iwx} & \text{for } a < x < b. \\ 0 & \text{otherwise.} \end{cases}$$

Soln We know that

$$\mathcal{F}[F(x)] = f(p) = \int_{-\infty}^{\infty} F(x)\, e^{ipx}\, dx$$

$$= \int_{-\infty}^{a} (0)\, dx + \int_{a}^{b} e^{i\omega x} e^{ipx}\, dx$$

$$+ \int_{b}^{\infty} 0\, dx$$

$$= \int_{a}^{b} e^{i(p+\omega)x}\, dx.$$

$$= \left[\frac{e^{i(p+\omega)x}}{i(p+\omega)}\right]_a^b$$

$$f(p) = \frac{1}{i(p+\omega)}\left[e^{i(p+\omega)b} - e^{i(p+\omega)a}\right]$$

Properties of fourier transform.

(i) Linear property or linearly :-
Let F and G be two function of class $L^1(-\infty,\infty)$ then

$$\mathcal{F}(aF + bG) = a\,\mathcal{F}(F) + b\,\mathcal{F}(G).$$

where a & b are constant.

Proof :- We have -

$$\mathcal{F}(aF(x) + bG(x)) = \int_{-\infty}^{\infty} (aF(x) + bG(x))\, e^{ipx}\, dx.$$

$$= \int_{-\infty}^{\infty} a\,F(x)\, e^{ipx}\, dx + \int_{-\infty}^{\infty} b\, G(x)\, e^{ipx}\, dx.$$

$= a\, \mathcal{F}[F(x)] + b\, \mathcal{F}[G(x)]$.

ii) <u>Change of Scale :-</u>

Let $f(p)$ be fourier transform of $F(x)$ then

$$\mathcal{F}[F(ax)] = \frac{1}{a} f\left(\frac{p}{a}\right). \qquad a>0.$$

Soln:-

$$\mathcal{F}[F(ax)] = \int_{-\infty}^{\infty} F(ax)\, e^{ipx}\, dx$$

$=$ Let $ax = t$

$a\,dx = dt$

$$= \int_{-\infty}^{\infty} F(t)\, e^{\frac{ipt}{a}} \frac{dt}{a}. \qquad \text{when } x \to -\infty,\ t \to -\infty$$

$$x \to \infty,\ t \to \infty$$

$$= \frac{1}{a} f\left(\frac{p}{a}\right).$$

iii) <u>Shifting property :-</u>

$$\mathcal{F}[F(x-a)] = e^{iap} f(p).$$

when $x \to -\infty$, $t \to -\infty$

$x \to \infty$, $t \to \infty$

Soln

$$\mathcal{F}[F(x-a)] = \int_{-\infty}^{\infty} F(x-a)\, e^{ipx}\, dx.$$

$$= \int_{-\infty}^{\infty} F(t)\, e^{ip(t+a)} \cdot dt \qquad x-a=t.$$

$$= e^{iap} \int_{-\infty}^{\infty} F(t)\, e^{i\cancel{a}t}\, dt. \qquad = e^{iap} f(p).$$

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* Modulation Theorem :-

• Let $f(p)$ be a fourier transform of $F(x)$ then

$$\mathcal{F}[F(x)\cos(ax)] = \frac{1}{2}\{f(p+a) + f(p-a)\}$$

Soln :-

$$\mathcal{F}[F(x)\cos(ax)] = \int_{-\infty}^{\infty} F(x)\cos(ax)\, e^{ipx}\, dx$$

we known that

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$= \int_{-\infty}^{\infty} F(x)\left[\frac{e^{iax} + e^{-ix.a}}{2}\right] e^{ipx}\, dx.$$

$$= \int_{-\infty}^{\infty} \frac{F(x)}{2}\left[e^{i(p+a)x} + e^{i(p-a)x}\right] dx.$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} F(x)\, e^{i(p+a)x}\, dx + \int_{-\infty}^{\infty} F(x)\, e^{i(p-a)x}\, dx.\right.$$

$$= \frac{1}{2}\left[f(p+a) + f(p-a)\right].$$

Let $f(p)$ be a fourier transform of $F(x)$.

$$\mathcal{F}[F(x)\sin(ax)] = \frac{1}{2i}\{f(p+a) - f(p-a)\}$$

Theorem:-

If $\mathcal{F}_s\{F(x)\} = f_s(p)$ and $\mathcal{F}_c[F(x)] = f_c(p)$

Then,

$$\mathcal{F}_s[x F(x)] = -\frac{d}{dp}[f_c(p)]$$

$$\mathcal{F}_c[x F(x)] = \frac{d}{dp}[f_s(p)]$$

<u>Proof:-</u>

$$f_c(p)\left[\text{i.e. } \mathcal{F}_c(F(x))\right] = \int_0^\infty F(x)\cos(px)\,dx$$

$$\frac{d}{dp} f_c(p) = \frac{d}{dp}\int_0^\infty F(x)\cos(px)\,dx.$$

$$= \int_0^\infty \left\{\frac{\partial}{\partial p} F(x)\cos(px)\right\} dx$$

$$= \int_0^\infty F(x)\left[-\sin(px)\cdot x\right] dx$$

$$= -\int_0^\infty \{x F(x)\}\sin(px)\,dx$$

$$= -\mathcal{F}_s[x F(x)].$$

$|x| < 1$

$\delta + a$

$\delta - a$

$|x - a| < \delta$

$|x - a| < \delta$

$x - a < \delta + a$

$x < \delta + a$

$-x + a < \delta$

$-x < \delta - a$

$-x < \delta + a$

Note:-

$$|x| = \begin{cases} x & x > 0 \\ -x & x < 0. \end{cases}$$

$F(x) = e^{-|x|}$

$$\int_{-\infty}^{\infty} F(x)\, e^{ipx}\, dx = \int_{-\infty}^{0} e^{-(-x)}\, e^{ipx}\, dx + \int_{0}^{\infty} e^{-x}\, e^{ipx}\, dx$$

$$= \int_{-\infty}^{0} e^{x} \cdot e^{ipx}\, dx$$

Q. (9) $F[e^{-x^2}]$. Hence find the fourier transform e^{-ax^2} $a > 0$.

i) e^{-ax^2} $a > 0$

ii) $e^{-\frac{x^2}{2}}$

iii) $e^{-4(x-3)}$

iv) $e^{-x^2} \cos 2x$.

Soln $F\{F(x)\} = \int_{-\infty}^{\infty} F(x)\, e^{ipx}\, dx = f(p)$

$$= \int_{-\infty}^{\infty} e^{-x^2}\, e^{ipx}\, dx.$$

$$= \int_{-\infty}^{\infty} e^{-x^2 + ipx}\, dx = \int_{-\infty}^{\infty} e^{x(ip - x)}\, dx.$$

$$= \int_{-\infty}^{\infty} e^{(ix)^2 + \frac{1}{2} i (\frac{p}{2})(x) + (\frac{p}{2})^2 - (\frac{p}{2})^2}\, dx$$

$$= \int_{-\infty}^{\infty} e^{(ix + \frac{p}{2})^2 - (\frac{p}{2})^2}\, dx.$$

$$= \int_{-\infty}^{\infty} e^{(ix + \frac{p}{2})^2} \cdot e^{-\frac{p^2}{4}}\, dx.$$

$$= e^{-\frac{p^2}{4}} \int_{-\infty}^{\infty} e^{(ix + \frac{p}{2})^2}\, dx$$

Let $z = ix + \frac{p}{2}$

$$= e^{-\frac{p^2}{4}} \int_{-\infty}^{\infty} e^{z^2}\, \frac{dz}{i}$$

$dz = i\, dx$

$$= \int_{-\infty}^{\infty} e^{-x^2}\, e^{ipx}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2 + ipx}\, dx = \cancel{\int_{-\infty}^{\infty} e^{-x^2 + \frac{2}{2} i}}$$

$$= \int_{-\infty}^{\infty} e^{[x^2 - \frac{ip}{2}]}$$

$$= \int_{-\infty}^{\infty}$$

$$F\{F(x)\} = \int_{-\infty}^{\infty} e^{-x^2} e^{ipx}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-[x^2 - pix]}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-[x^2 + (\frac{pi}{2})^2 - pix - (\frac{pi}{2})^2]}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-[(x - \frac{pi}{2})^2 + \frac{p^2}{4}]}\, dx$$

$$= e^{-\frac{p^2}{4}} \int_{-\infty}^{\infty} e^{-(x - \frac{pi}{2})^2}\, dx$$

$x - \frac{ip}{2} = z$

$dx = dz$

$$= \int_{-\infty}^{\infty} e^{-z^2}\, dz$$

$$= e^{-\frac{p^2}{4}} \sqrt{\pi}.$$

(ii) e^{-ax^2}

$$L\{f(ax)\} = \frac{1}{a'} f\left(\frac{p}{a'}\right)$$

$$= \frac{1}{\sqrt{a}} f\left(\frac{p}{\sqrt{a}}\right)$$

$$= \frac{1}{a} e^{-\frac{p^2}{4a}} \sqrt{\pi}.$$

(ii) $e^{-\frac{x^2}{2}}$

$$L\{f(ax)\} = \frac{1}{(\frac{1}{2})} e^{-\frac{2p^2}{4}} \sqrt{\pi}$$

$$= 2 e^{-\frac{p^2}{2}} \sqrt{\pi}.$$

(iii) $e^{-4(x-3)^2}$

Let $L\{f(x-3)\} = \frac{1}{4} f\left(\frac{p}{4}\right)$.

$$= \frac{1}{2} f\left(\frac{p}{2}\right)$$

$$= \frac{1}{2} e^{\frac{3ip}{2}} e^{-(\frac{p}{2})^2 \cdot \frac{1}{4}} \sqrt{\pi}$$

$$L\{f(x-3)\} = e^{i3p} f(p)$$

$$= e^{3ip} e^{-\frac{p^2}{4}} \sqrt{\pi}.$$

$$= \frac{1}{2} e^{\frac{3ip}{4}} e^{-\frac{p^2}{16}} \sqrt{\pi}$$

(iv) $e^{-x^2} \cos 2x$

Let

$$L\{e^{-x^2} \cos 2x\}$$

$$= \frac{1}{2}[f(p+a) + f(p-a)].$$

$$= \frac{1}{2}[f(p+2) + f(p-2)]$$

$$= \frac{1}{2}\left[e^{-\frac{(p+2)^2}{4}} \sqrt{\pi} + e^{-\frac{(p-2)^2}{4}} \sqrt{\pi}\right].$$

Q. find the fourier transform of

$$F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a. \end{cases}$$

Hence evaluate (i) $\int_{-\infty}^{\infty} \frac{\sin(ap)\cos px}{p}\, dp.$

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(ii) $\int_0^\infty \frac{\sin p}{p}\, dp$.

Soln We know that

$$\mathcal{F}\{F(x)\} = \int_{-\infty}^{\infty} F(x)\, e^{ipx}\, dx = f(p).$$

Now, $\mathcal{F}\{F(x)\} = \int_{-\infty}^{-a} 0 + \int_{-a}^{a} 1\, e^{ipx} dx + \int_a^\infty 0\, e^{ipx} dx$

$$= \int_{-a}^{a} e^{ipx}\, dx$$

$$= \left[\frac{e^{ipx}}{(ip)}\right]_{-a}^{a}$$

$$= \frac{1}{ip}\left[e^{iap} - e^{-iap}\right].$$

$$= \frac{1}{ip}\left[2i \sin(ap)\right]$$

$$= \frac{2}{p}\sin(ap) = f(p). \quad \text{(say)} \quad \text{(i)}$$

Now

$\cancel{\int_{-\infty}^{\infty} \frac{\sin(ap)\cos(px)}{p}\, dP =}$

$$F^{-1}\left\{\frac{2}{p}\sin(ap)\right\} = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{p}\sin(ap)\, e^{-ipx}\, dp = F(x)$$

$$\Rightarrow \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\cancel{2}}{p}\sin(ap)\,(\cos px - i\sin px)\, dp$$

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$$= \frac{1}{\pi}\left[\int_{-\infty}^{\infty} \frac{\sin(ap)\cos px}{p}\, dP - \int_{-\infty}^{\infty} \frac{\sin(ap)}{p}\sin(px)\, dP\right]$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{\infty} \frac{\sin(ap)\cos(px)}{p}\, dP\right] \quad \text{odd function}$$

$$= f(x)$$

$$\int_{-\infty}^{\infty} \frac{\sin(ap)\cos(px)}{p}\, dP = \pi \begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases}$$

ii) $\int_0^\infty \frac{\sin p}{p}\, dp$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin(ap)\cos(px)}{p}\, dp = \begin{cases} \pi & |x| \leq a \\ 0 & |x| \geq a \end{cases}$$

$$= \int_0^\infty \frac{\sin(ap)\cos(px)}{p}\, dp = \begin{cases} \frac{\pi}{2} \\ 0 \end{cases}$$

put $a = 1$ $\quad x = 0$

$|0| < 1$

$\therefore \; = \frac{\pi}{2}.$

Q. $F(x) = \begin{cases} 1-x^2 & \text{if } |x|<1. \\ 0 & \text{if } |x|>1. \end{cases}$

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$$\mathcal{F}\{F(x)\} = \int_{-\infty}^{\infty} F(x)\, e^{ipx}\, dx.$$

$$= \int_{-1}^{1} (1-x^2)\, e^{ipx}\, dx.$$

$$= \int_{-1}^{1} (1-x^2) \left(\frac{\cos px - i\sin px}{\cancel{2}}\right) dx.$$

$$= \frac{2}{\cancel{2}} \int_0^1 (1-x^2)(\cos px - \underbrace{i\sin px}_{\text{zero}})\, dx.$$

$$= 2\int_0^1 (1-x^2)\, (\cos px - \cancel{i\sin px})\, dx$$

$$= 2\int_0^1 (1-x^2)\cos px\, dx.$$

$$= \cancel{2\int_0^1 \cos px\, dx} - \cancel{\int_0^1 x^2 \cos px\, dx}$$

$$= 2(1-x^2)\int_0^1 \cos px\, dx - \int (-2x)\int_0^1 \cos px\, dx$$

$$= 2(1-x^2)\left[\frac{\sin(px)}{(p)}\right]_0^1 + 2\int x \frac{\sin(px)}{p}\, dx.$$

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$$\mathcal{F}\{f(x)\} = \left\{\frac{4}{p^3}(\sin p - p\cos p)\right\} = f(p) \text{ say.}$$

$$\mathcal{F}^{-1}\{f(p)\} = F(x)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{p^3}(\sin p - p\cos p)\, e^{ipx}\, dx = F(x).$$

Leibnitz theorem

$$\frac{d}{dx}\int_{G(x)}^{H(x)} f(x,t)\, dt = \int_{G(x)}^{H(x)} \frac{\partial}{\partial x} f(x,t)\, dt$$

$$+ f(x, H(x))\, \frac{\partial H(x)}{\partial x}$$

$$- f(x, G(x))\, \frac{\partial G(x)}{\partial x}.$$

When $H(x)$ & $G(x)$ are constant

$$\boxed{\frac{d}{dx}\int_{G(x)}^{H(x)} f(x,t)\, dt = \int_{G(x)}^{H(x)} \frac{\partial}{\partial x} f(x,t)\, dt}.$$

Convolution Theorem

A convolution is an integral that expresses the amount of overlap of a function say $f_1(t)$ when it is shifted over another function, say $f_2(t)$ it blends one function with.

Mathematically, the convolution of two function $f(x)$ and $G(x)$ defined in $-\infty < x < \infty$ is defined as -

$$F(x) * G(x) = \int_{u=-\infty}^{u=\infty} F(u) \cdot G(x-u)\, du.$$

Fourier transform of convolution of two function is the product of their individual fourier transform i.e. If $f(x)$ and $G(x)$ be two function in $-\infty < x < \infty$ then

$$\mathcal{F}\{F(x) * G(x)\} = \mathcal{F}\{F(x)\} * \mathcal{F}\{G(x)\}$$

Q.

Convulation theorem

Proof :- $\mathcal{F}\{F(x) * G(x)\} = \mathcal{F}\left\{\int_{u=-\infty}^{\infty} F(u) \cdot G(x-u)\, du\right\}$

We know that

$$\mathcal{F}\{F(x) * G(x)\} = \mathcal{F}\left\{\int_{-\infty}^{\infty} F(u) \cdot G(x-u)\, du\right\}$$

$x-u=t$
$x=t+u$

$$= \mathcal{F}\left\{\int_{-\infty}^{\infty} F(u) \cdot G(x-u)\, du\right\}$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} F(u) \cdot G(x-u)\, du\right\} e^{ipx}\, dx$$

$$= \int_{-\infty}^{\infty} F(u)\left\{\int_{-\infty}^{\infty} G(x-u) \cdot e^{ipx}\, dx\right\} du$$

$$= \int_{-\infty}^{\infty} F(u)\left\{\int_{-\infty}^{\infty} G(t) \cdot e^{ip(t+u)}\, dt\right\} du$$

$x-u=t$
$x=t+u$
$dx=dt$

$$= \int_{-\infty}^{\infty} F(u)\left\{\int_{-\infty}^{\infty} G(t) \cdot e^{ip(t+u)}\, dt\right\} du$$

$$= \int_{u=-\infty}^{\infty} F(u) \cdot e^{ipu}\, du \left\{\int_{t=-\infty}^{\infty} G(t) \cdot e^{ipt}\, dt\right\}$$

$$= \mathcal{F}\{F(u)\} \cdot \mathcal{F}\{G(t)\}$$

$$= \mathcal{F}\{F(x)\} \cdot \mathcal{F}\{G(x)\}$$

Ques:- A linear system has a pulse $F(x)$ and is subjected to a rectangular pulse $g(x)$ as given below —

$$F(x) = \begin{cases} 0 & x < 0 \\ a e^{-ax} & x > 0 \end{cases}$$

$$G(x) = \begin{cases} 0 & -\infty < x < -1 \\ 1 & -1 < x < 1 \\ 0 & x > 1 \end{cases}$$

Find the output time function by convolving F and G.

Soln:

Application of fourier transform to partial differential Eqn :-

Some important result:-

If $\mathcal{F}\{u(x,t)\} = \int_{-\infty}^{\infty} u(x,t)\, e^{ipx}\, dx = \bar{u}(p,t)$

[Fourier transform of $u(x,t)$ w.r.t. 'x']

then,

Proof:- We have

$$\mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{ipx}\, dx.$$

$$= \underbrace{e^{ipx} \frac{\partial u}{\partial x}\Big]_{-\infty}^{\infty}}_{0} - ip\int_{-\infty}^{\infty} e^{ipx} \frac{\partial u}{\partial x}\, dx.$$

$$= 0 - ip\int_{-\infty}^{\infty} e^{ipx} \frac{\partial u}{\partial x}\, dx$$

$$= -ip\left[\left(e^{ipx}(u)\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{ipx}(ip)u\, dx.\right]$$

$$= -ip\left[0 - \int_{-\infty}^{\infty} (ip)e^{ipx} u\, dx\right]$$

$$= -p^2 \int_{-\infty}^{\infty} e^{ipx} u\, dx. = -p^2 \int_{-\infty}^{\infty} u(x,t) e^{ipx}\, dx$$

$$\boxed{\mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -p^2\, \bar{u}(p,t)}$$

In general,

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (-ip)^n\, \bar{u}(p,t).$$

$\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} =$

Then $\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}\, e^{ipx}\, dx.$

$= \frac{d}{dt}\left[\int_{-\infty}^{\infty} u(x,t)\, e^{ipx}\, dx\right]$

$$\boxed{\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\, \bar{u}(p,t).}$$

Similarly.
$$\boxed{F\left\{\frac{\partial^2 u}{\partial t^2}\right\} = \frac{d^2}{dt^2}\, \bar{u}(p,t)}$$

* Fourier Sine Transform * of $u(x,t)$ w.r.t. 'x'

$\mathcal{F}_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^{\infty} \underset{\text{II}}{\frac{\partial^2 u}{\partial x^2}}\, \underset{\text{I}}{\sin px}\, dx$

$= \sin px\, \frac{\partial u}{\partial x}\Big]_0^{\infty} - \int_0^{\infty} \cos px\,(p)\, \frac{\partial u}{\partial x}\, dx$

$= 0 - p\int_0^{\infty} \cos px\, \frac{\partial u}{\partial x}\, dx.$

$= -p\left[(\cos px\ u)_0^{\infty} - \int_0^{\infty} \sin px\,(-p)\, u\, dx\right]$

$= -p\left[(0 - u(0,t)) + p\int_0^{\infty} u(x,t)\, \sin px\, dx\right]$

Assuming $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$.

$$\boxed{\mathcal{F}_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -p^2\, \bar{u}_s(p,t) + p\, u(0,t)}$$

$\mathcal{F}_s\left\{\frac{\partial u}{\partial t}\right\} = \int_0^{\infty} \frac{\partial u}{\partial t}\, \sin px\, dx.$

$\mathcal{F}_s\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\int_0^{\infty} u(x,t)\, \sin px\, dx.$

$$\boxed{\mathcal{F}_s\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\, \bar{u}_s(p,t)}.$$

Similarly,
$\mathcal{F}_s\left\{\frac{\partial^2 u}{\partial t^2}\right\} = \frac{d^2}{dt^2}\, \bar{u}_s(p,t)$

Fourier Cosine Transform :-

If $\mathcal{F}_c\{u(x,t)\} = \int_0^{\infty} u(x,t)\cos px\, dx. = \bar{u}_c(p,t)$

[Fourier cosine transform of $u(x,t)$ w.r.t. x.

$\mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}(x,t)\right\} = \int_0^{\infty} \frac{\partial^2 u}{\partial x^2}(x,t)\cos px\, dx$

$= \cos px \cdot \frac{\partial u}{\partial x}\Big]_0^{\infty} - \int_0^{\infty} -\sin px\,(p)\, \frac{\partial u}{\partial x}\, dx.$

$= \left[0 - \left(\frac{\partial u}{\partial x}\right)_{x=0}\right] + p\int_0^{\infty} \frac{\partial u}{\partial x}\, \sin px\, dx$

$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + p\left[\sin px\ u\Big]_0^{\infty} - \int_0^{\infty} \cos px\,(p)\cdot u\, dx\right]$

$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + p\left[(0-0) - p\int_0^{\infty} u(x,t)\cos px\, dx\right]$

$$= \left(-\frac{\partial u}{\partial x}\right)_{x=0} - p^2 \int_0^\infty u(x,t)\cos px\,dx.$$

$$\boxed{F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\left(\frac{\partial u}{\partial x}\right)_{x=0} - p^2\,\bar{u}_c(p,t)}$$

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = \int_0^\infty \frac{\partial u}{\partial t}\cos px\,dx.$$

$$= \frac{d}{dt}\int_0^\infty u(x,t)\cos px\,dx.$$

$$\boxed{F_c\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\,\bar{u}_c(p,t).}$$

Similarly,

$$\boxed{F_c\left\{\frac{\partial^2 u}{\partial t^2}\right\} = \frac{d^2}{dt^2}\,\bar{u}_c(p,t)}$$

Note:- Application of fourier transform to partial diffrential Equation.

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

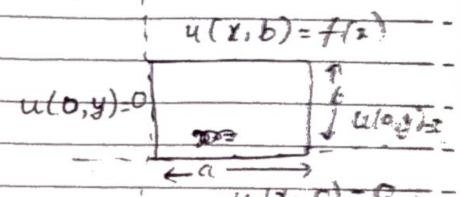
$u(0,t)=0$, $t>0$
$u(l,t)=0$, $t>0$
$u(x,0)=f(x)$, $0<x<l$
$\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x)$

(2) $$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

$u(0,t)=A$ $t>0$
$u(l,t)=B$ $t>0$
$u(x,0)=f(x)$

(3) heat Eqn under Steady State
laplace Eqn

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$$\left.\begin{aligned} u(x,0)&=0\\ u(x,b)&=f(x)\\ u(0,y)&=0\\ u(a,y)&=0 \end{aligned}\right\}$$

All these problem are being ~~should~~ solved in finite region allowing us to ~~utif~~ utilize the boundary condition while intending to solve problems in infinite region, we discard the boundary condition in favour of boundness at infinity.

final note:-
The type of boundary condition determine which of fourier sine or cosine transform is to be used.
If boundary condition of Dirchlets type type: (w/o derivative)
↳ sine transform is to be used

While cosine transform is to be used if the boundary condition is of Neumann's type (with derivative)

Q. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ $\quad 0<x<\infty, \; t>0$ —— (1)

$u(0,t) = A \quad (A \neq 0)$ —— (2)
$u(x,0) = f(x)$ —— (3)

Soln:- We have $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad 0<x<\infty$

Applying fourier sine transform, we get.

$$F_s\left\{\frac{\partial u}{\partial t}\right\} = F_s\left\{c^2 \frac{\partial^2 u}{\partial x^2}\right\}$$

$$F_s\left\{\frac{\partial u}{\partial t}\right\} = c^2 \left\{F_s\left(\frac{\partial^2 u}{\partial x^2}\right)\right\}$$

$$F_s\left\{\frac{\partial u}{\partial t}\right\} = c^2 F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\cancel{F_s\left\{\frac{\partial u}{\partial t}\right\}} = \frac{d}{dt}\bar{u}_s(p,t) = c^2\left\{p\cdot u(0,t) - p^2 \bar{u}_s(p,t)\right\}$$

$$\frac{d}{dt}\bar{u}_s(p,t) \mp p^2c^2\bar{u}_s(p,t) = c^2 p\, u(0,t).$$

$$\frac{d}{dt}\bar{u}_s(p,t) + p^2c^2\bar{u}_s(p,t) = pc^2 u(0,t).$$
$$= pc^2 A \quad \text{—— (4)}$$

This is linear ~~PDE~~ ODE with first order

So, $I.F = e^{\int p^2c^2 dt} = e^{p^2c^2t}$.

Soln is given by –

$$\bar{u}_s \times e^{p^2c^2t} = \int e^{p^2c^2t}\cdot pc^2 A\, dt$$

$$\bar{u}_s(p,t)\, e^{p^2c^2t} = \frac{e^{p^2c^2t}}{(p^2c^2)} pc^2 A + C_1$$

$$\bar{u}_s(p,t)\, e^{p^2c^2t} = \frac{e^{p^2c^2t}}{(p^2c^2)}(pc^2A) + C_1$$

—— (5)

Now from (3)

$$u(x,0) = f(x)$$

$$F_s\{u(x,0)\} = \bar{f}(p)$$

$$\bar{u}_s(p,0) = \bar{f}(p) \quad \text{--- (6)}$$

Now from (5)

$$\bar{u}_s(p,t)\, e^{p^2c^2t} = c^2 p A \cdot \frac{e^{p^2c^2t}}{p^2c^2} + C_1$$

at $t=0$ $$\bar{u}_s(p,0) = \frac{A}{p} + C_1 \quad (\text{from } (6))$$

$$\bar{f}(p) = \frac{A}{p} + C_1$$

$$C_1 = \bar{f}(p) - \frac{A}{p}$$

So (5) gives

$$\bar{u}_s(p,t)\, e^{p^2c^2t} = \frac{A}{p} e^{p^2c^2t} + \bar{f}(p) - \frac{A}{p}$$

$$F_s\{u(x,t)\} = \frac{A}{p} + \left\{\bar{f}(p) - \frac{A}{p}\right\} e^{-p^2c^2t}$$

$$u(x,t) = F_s^{-1}\left\{\frac{A}{p} + \left\{\bar{f}(p) - \frac{A}{p}\right\} e^{-p^2c^2t}\right\}$$

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{A}{p} + \bar{f}(p) - \frac{A}{p}\, e^{-p^2c^2t}\right) \sin px \, dp$$